Problem Set 8 – Relativity (`GOI36A`)

10/11/2020

1 Free Scalar Fields in Curved Spacetime

At the beginning of the semester, we talked about defining an action S for a scalar field by

$$S = \int d^4x \, \mathcal{L} \; , \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian, which for a free scalar of mass m in Minkowski spacetime

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \; . \tag{1.2}$$

We can then use the Euler-Lagrange equations to derive an equation of motion for the field ϕ and study its motion.

What we want to do now is reconsider this action when the scalar moves on a curved spacetime instead of on a flat Minkowski spacetime. The way we will do this is through *minimal coupling*, in which we take the same form of the action but we change just enough to make it covariant with respect to an arbitrary curved background. In particular, there are three things we need to change:

- The Minkowski metric $\eta_{\mu\nu}$ must be replaced with an arbitrary metric $g_{\mu\nu}$, which in principle also depends on the spacetime coordinate x^μ.
- Ordinary partial derivatives ∂_μ should be replaced with covariant derivatives ∇_μ. This doesn't at first appear to make a big difference for scalar fields, since $\partial_\mu\phi = \nabla_\mu\phi$, but this will matter when deriving the equations of motion.
- The integration measure d^4x is not a good measure in curved spacetimes, since it will transform under coordinate transformations. We need to fix this by adding in the Jacobian factor $\sqrt{-g}$ to account for coordinate transformations:

  $$d^4x \to d^4x\,\sqrt{-g} \; . \tag{1.3}$$

  This new integration measure is invariant under coordinate transformations. (It's a good exercise to prove this fact!)

With these three steps in mind, the action for a free scalar field becomes

$$S = \int d^4x\,\sqrt{-g}\,\mathcal{L} \; , \quad \mathcal{L} = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 \; , \tag{1.4}$$

where the metric tensor $g_{\mu\nu}$ and the scalar field ϕ are both understood to be functions of the spacetime coordinate x^μ.

1.) The Euler-Lagrange equations for the scalar field ϕ take the form

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)}\right) = 0 \; . \tag{1.5}$$

Show that these Euler-Lagrange equations for our scalar field theory result in the Klein-Gordon equation:

$$(\Box - m^2)\phi = 0 \; , \tag{1.6}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$ is the covariant d'Alembertian, but is often just referred to as the Laplacian.

2.) To understand how to solve this equation, it would be helpful to express the Laplacian in terms of ordinary derivatives in an explicit coordinate frame, since this will let us actually write down concrete expressions. Your task is to now prove that

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) \ . \tag{1.7}$$

As an intermediate step, you may find it very helpful to first prove that

$$\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \Gamma^{\nu}_{\mu\nu} \ , \tag{1.8}$$

and then use the product rule in (1.7) to expand the right-hand side in terms of ordinary derivatives and Christoffel symbols. These terms will combine in such a way to form a covariant derivative, leaving you with $\nabla_\mu\nabla^\mu\phi$.

Throughout this problem, you will need to use *Jacobi's formula*, which says that given any linear operator D and any invertible matrix M, they satisfy

$$D\det(M) = \det(M)\,\mathrm{tr}(M^{-1}DM) \ . \tag{1.9}$$

3.) Armed with (1.7), we can now study what the Klein-Gordon equation is on explicit backgrounds. In particular, let's consider the Minkowski metric in spherical coordinates:

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\psi^2 \ , \tag{1.10}$$

where we are using ψ as our azimuthal coordinate, since ϕ is already being used for the scalar field.

Write down the Klein-Gordon equation explicitly in spherical coordinates. How does this compare to what the Klein-Gordon equation looks like in the usual Cartesian coordinates?

4.) Write down the Klein-Gordon equation for a free scalar when the background is the Schwarzschild spacetime. It will be somewhat similar to your result from the previous problem, but with some modifications. Feel free to set $c = 1$ if you want to make your life easier on this one.

At this point, you've written down everything one would need to completely describe the dynamics of a scalar field on a Schwarzschild spacetime. This is a cool result, and you should appreciate how far you've come in this semester to get to this point. From here, generalizing this process to other types of particles and fields is straightforward.

2 The Einstein Equation with a Free Scalar

One thing to note is that we have so far treated this scalar as a *probe particle*, meaning that we have assumed that we could place it on a curved background without affecting the dynamics of the metric tensor itself. Such approximations are often used in theoretical physics, but the more general approach is to consider both the metric tensor $g_{\mu\nu}$ and the scalar field ϕ as dynamical fields.

That is, the full action should actually be given by

$$S = S_{\mathrm{EH}} + S_{\mathrm{M}} \ , \tag{2.1}$$

where $S_{\rm EH}$ is the Einstein-Hilbert action

$$S_{\rm EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R \ , \tag{2.2}$$

while $S_{\rm M} = \int d^4x \sqrt{-g}\, \mathcal{L}_{\rm M}$ is the action for all other matter fields in the theory. For a free scalar, this matter action is simply the one we looked at in (1.4), and so our full action is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G_N} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) \ . \tag{2.3}$$

There are two associated Euler-Lagrange equations for this action; one for the metric tensor $g_{\mu\nu}$, and one for the scalar field ϕ. Since the Einstein-Hilbert term doesn't affect how we vary S with respect to the field ϕ, the scalar field equation of motion will still be the Klein-Gordon equation. And, the equation of motion for $g_{\mu\nu}$ is the Einstein equation, as you have seen by now.

Thus, the equations of motion for our theory with a scalar coupled to gravity are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} &= 8\pi G_N T_{\mu\nu} \ , \\ (\Box - m^2)\phi &= 0 \ , \end{aligned} \tag{2.4}$$

where $T_{\mu\nu}$ is the stress-energy tensor, which is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\, \mathcal{L}_{\rm M})}{\delta g^{\mu\nu}} \ . \tag{2.5}$$

Note that these equations of motion have to be solved simultaneously. So, in general it is not enough to simply claim that we can put a field on any curved background we want; we have to show that the full equations are compatible with this.

5.) Derive the stress-energy tensor $T_{\mu\nu}$ for the free scalar field. Show that it is conserved:

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{2.6}$$

6.) Convince yourself that there is a trivial way to allow for a Schwarzschild solution to the Einstein equation by setting $\phi = 0$.

7.) Without actually computing anything, try to think about what would happen if we added a term to our action that looked like

$$S' = \int d^4x \sqrt{-g}\, \phi^2 R \ . \tag{2.7}$$

What would this mean for the equations of motion? Would the Einstein equation be modified? Would the scalar equation of motion be modified?

Finding solutions to the equations of motion when matter fields are turned on is not always easy. They require solving some pretty awful-looking second-order partial differential equations. Luckily, we live in the age where you can attack these things with computers! There is a rich literature on numerically solving the Einstein equation when matter fields (like scalars) are around. If you're curious about playing around with some of the software yourself, ask me to point you in the right direction!